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Estimating the Lieb-Robinson Velocity for Classical Anharmonic Lattice Systems

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Abstract We estimate the Lieb-Robsinon velocity, also known as the group velocity, for a system of harmonic oscillators and a variety of anharmonic perturbations with mainly short-range interactions. Such bounds demonstrate a quasi-locality of the dynamics in the sense that the support of the time evolution of a local observable remains essentially local. Our anharmonic estimates are applicable to a special class of observables, the Weyl functions, and the bounds which follow are not only independent of the volume but also the initial condition.

Keywords Lieb-Robinson · Locality bounds · Classical dynamics · Anharmonic · Group velocity

1 Introduction

A notion of locality is crucial in rigorously analyzing most physical systems. Typically, sets of local observables are associated with bounded regions of space, and one is interested in how these observables evolve dynamically with respect to the interactions governing the system. In relativistic theories, the evolution of a local observable remains local, i.e. the support of dynamically evolved local observables is restricted to a light cone. For non-relativistic models, such as those we will be considering in the present work, the dynamics does not preserve locality in the sense that, generically, an observable initially chosen localized to a particular site is immediately evolved into an observable dependent on all sites of the system.

In 1972, Lieb and Robinson [8] explored a quasi-locality of the dynamics corresponding to non-relativistic quantum spin systems. Roughly speaking, a quantum spin system is de-

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scribed by a self-adjoint Hamiltonian, which describes the interactions of the system, and its associated Heisenberg dynamics, see e.g. [2] for more details. The estimates they proved demonstrate that, up to exponentially small errors, the time evolution of a local observable remains essentially supported in a ball of radius proportional to $v|t|$ for some $v > 0$. This quantity v, which we describe as a Lieb-Robinson velocity, defines a natural rate of propagation, and it can be estimated in terms of the system's free parameters, for example, the interaction strength of the Hamiltonian.

The models analyzed in this paper will correspond to a classical system of oscillators evolving according to a Hamiltonian dynamics. Hamiltonians of this type have frequently appeared in the literature as their analysis provides an important means of studying the emergence of non-equilibrium phenomena in macroscopic systems. For example, rigorous results on the existence of the thermodynamic limit for these models date back to [9]. A notion of quasi-locality, similar to the Lieb-Robinson bounds mentioned above, for these classical oscillator systems was originally considered in 1978 by Marchioro et al. in [15]. More recently, a nice generalization of the estimates found in [15], those pertaining to models with, e.g. a quartic on-site term, appeared in [4]. Both of these results were obtained in the spirit of deriving a classical analogue of the Lieb-Robinson bounds found in [8].

Over the past few years a number of important improvements on the original Lieb-Robinson bounds have appeared in the literature [5, 6, 10–12], see [14] for the most current review article. These new estimates have found a variety of intriguing applications [3, 5, 7, 13], but perhaps most interestingly for the present work, the results found in [11] establish bounds which are applicable beyond the context of quantum spin systems. In [11], the authors prove a version of the Lieb-Robinson bounds for quantum anharmonic lattice systems. Motivated by [11], the main goal of this paper is to employ these new methods to establish explicit bounds on the Lieb-Robinson velocity for a large class of anharmonic lattice systems.

To express our results more precisely, we introduce the following notation. We will consider systems confined to a large but finite subset $\Lambda \subset \mathbb{Z}^\nu$; here $\nu \geq 1$ is an integer. With each site $x \in \Lambda$, we will associate an oscillator with coordinate $q_x \in \mathbb{R}$ and momentum $p_x \in \mathbb{R}$. The state of the system in Λ will be described by a sequence $\mathrm{x} = \{(q_x, p_x)\}_{x\in\Lambda}$, and phase space, i.e. the set of all such sequences, will be denoted by $\mathcal{X}_\Lambda$.

A Hamiltonian, H, is a real-valued function on phase space. Typically the Hamiltonian of interest generates a flow, Φ_t, on phase space. Specifically, given $H : \mathcal{X}_\Lambda \to \mathbb{R}$ one defines, for any $t \in \mathbb{R}$, a function $\Phi_t : \mathcal{X}_\Lambda \to \mathcal{X}_\Lambda$ by setting $\Phi_t(\mathrm{x}) = \{(q_x(t), p_x(t))\}_{x\in\Lambda}$, the sequence whose components satisfy Hamilton's equations: for any $x \in \Lambda$,

$$\begin{aligned} \dot{q}_x(t) &= \frac{\partial H}{\partial p_x}(\Phi_t(\mathrm{x})), \\ \dot{p}_x(t) &= -\frac{\partial H}{\partial q_x}(\Phi_t(\mathrm{x})), \end{aligned} \tag{1.1}$$

with initial condition $\Phi_0(\mathrm{x}) = \mathrm{x}$.

To measure the effects of this Hamiltonian dynamics on the system, one introduces observables. An observable A is a complex-valued function of phase space. We will denote by $\mathcal{A}_\Lambda$ the space of all local observables in Λ, i.e. the set of all functions $A : \mathcal{X}_\Lambda \to \mathbb{C}$. A given Hamiltonian, H, generates a dynamics α_t on the space of local observables in the sense that, for any $t \in \mathbb{R}$, the dynamics $\alpha_t : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ is defined by setting $\alpha_t(A) = A \circ \Phi_t$.

For the locality result we will present, the notion of support of a local observable is important. Given $A \in \mathcal{A}_\Lambda$, the support of A is defined to be the minimal set $X \subset \Lambda$ for which A depends only on those parameters q_x or p_x with $x \in X$.

In classical systems evolving with a Hamiltonian dynamics, quasi-locality can be expressed in terms of the Poisson bracket between local observables. Here the Poisson bracket is the observable given by

$$\{A, B\} = \sum_{x \in \Lambda} \frac{\partial A}{\partial q_x} \cdot \frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x} \cdot \frac{\partial B}{\partial q_x}, \tag{1.2}$$

for sufficiently smooth observables A and B. Observe that for disjoint subsets $X, Y \subset \Lambda$ and observables A with support in X and B with support in Y, it is clear that $\{A, B\} = 0$. The quasi-locality question of interest in this context is: given a Hamiltonian H, its corresponding dynamics α_t, and a pair of observables A and B with disjoint supports, does the quantity $\{\alpha_t(A), B\}$ satisfy an estimate, which decays in the distance between the supports of A and B, for small times t? Note that $\{\alpha_t(A), B\}|_{t=0} = \{A, B\} = 0$. We say that a Hamiltonian H has a finite Lieb-Robinson velocity, if for some $\mu > 0$ there exists $v > 0$ for which an estimate of the form

$$|\{\alpha_t(A), B\}(\mathrm{x})| \le Ce^{-\mu(d(X,Y)-v|t|)}, \tag{1.3}$$

holds for a class of local observables A and B and t sufficiently small. Here $d(X, Y)$ denotes the distance between the supports of the local observables A and B. This bound demonstrates that for times t with $|t| \le d(X, Y)/v$, the Poisson bracket remains exponentially small. With H and μ fixed, the infimum over all $v > 0$ for which (1.3) holds is the system's Lieb-Robinson velocity. The main goal of this paper is to provide estimates on this quantity for a variety of models. In proving bounds of the form (1.3), special attention must be given to the dependence of the numbers C and v on the observables A and B, the initial condition x, and the free parameters in the Hamiltonian. Most crucially, these numbers must be independent of the underlying volume Λ, so that they persist in the thermodynamic limit; once the existence of such a limit has been established.

We begin our analysis by considering finite volume restrictions of the harmonic Hamiltonian, i.e. $H_h^\Lambda : \mathcal{X}_\Lambda \to \mathbb{R}$ is given by

$$H_h^\Lambda(\mathrm{x}) = \sum_{x \in \Lambda} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{\nu} \lambda_j (q_x - q_{x+e_j})^2, \tag{1.4}$$

where e_j, for $j = 1, \ldots, \nu$, are the canonical basis vectors in $\mathbb{Z}^\nu$, and the parameters $\omega \ge 0$ and $\lambda_j \ge 0$ are the on-site and coupling strength, respectively. As is well-known, a variety of explicit calculations may be performed for this harmonic Hamiltonian. Perhaps most importantly, for any integer $L \ge 1$ and each subset $\Lambda_L = (-L, L]^\nu \subset \mathbb{Z}^\nu$, the flow $\Phi_t^{h,L}$ corresponding to $H_h^{\Lambda_L}$ may be explicitly computed, see Sect. 2.1 for details. Once the flow is known, a locality estimate easily follows for a specific set of observables.

We will equip the set of local observables $\mathcal{A}_{\Lambda_L}$ with the sup-norm, and we will say that $A \in \mathcal{A}_{\Lambda_L}$ is bounded if

$$\|A\|_\infty = \sup_{\mathrm{x} \in \mathcal{X}_{\Lambda_L}} |A(\mathrm{x})| \tag{1.5}$$

is finite. Furthermore, we will denote by $\mathcal{A}_{\Lambda_L}^{(1)}$ the set of all $A \in \mathcal{A}_{\Lambda_L}$ for which: given any $x \in \Lambda_L$, $\frac{\partial A}{\partial q_x} \in \mathcal{A}_{\Lambda_L}$, $\frac{\partial A}{\partial p_x} \in \mathcal{A}_{\Lambda_L}$, and

$$\|\partial A\|_\infty = \sup_{x \in \Lambda_L} \max\left(\left\| \frac{\partial A}{\partial q_x} \right\|_\infty, \left\| \frac{\partial A}{\partial p_x} \right\|_\infty \right) < \infty. \tag{1.6}$$

We can now state our first result.

Theorem 1 *Let X and Y be finite subsets of $\mathbb{Z}^\nu$ and take L_0 to be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$, denote by $\alpha_t^{h,L}$ the dynamics corresponding to $H_h^{\Lambda_L}$. For any $\mu > 0$ and any observables $A, B \in \mathcal{A}_{\Lambda_{L_0}}^{(1)}$ with support of A in X and support of B in Y, there exist positive numbers C and v_h, both independent of L, such that the bound*

$$\left\| \left\{ \alpha_t^{h,L}(A), B \right\} \right\|_\infty \leq C \|\partial A\|_\infty \|\partial B\|_\infty \min(|X|, |Y|) e^{-\mu(d(X,Y) - v_h|t|)} \tag{1.7}$$

holds for all $t \in \mathbb{R}$.

Some additional comments are in order. First, the quantity $d(X, Y)$ appearing above denotes the distance between the sets X and Y, measured in the L^1-sense, and for any $Z \subset \Lambda_L$, the number $|Z|$ is the cardinality of Z. Next, the fact that the bound (1.7) is true for any $\mu > 0$ implies that the Poisson bracket above has arbitrarily fast exponential decay in space. To achieve faster decay in space, however, the number C and the harmonic Lieb-Robinson velocity v_h increase. We describe our optimal bound on the harmonic velocity $v_h(\mu)$ in Sect. 2.2. Lastly, it is important to note that the quantities C and v_h are not only independent of the length scale L, they are also independent of the initial condition $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$.

An analogue of Theorem 1 appears already in [15]. In fact, for a specific one-dimensional system it is shown in [15] that an estimate of the form (1.7) follows from known results for Bessel functions; see also comments prior to Theorem 3 in Sect. 2.2. For our more general harmonic interactions in multi-dimensions, a similar analysis applies. This is the content of Sect. 2. Moreover, to prepare for our perturbative analysis in Sects. 3 and 4, we also provide explicit estimates on the harmonic Lieb-Robinson velocity, v_h, in terms of the system's free parameters $\omega \geq 0$ and $\lambda_i \geq 0$; see (2.30) and the comments following.

Our next result, Theorem 2 below, concerns on-site perturbations of the harmonic Hamiltonian. To state this precisely, fix a function $V : \mathbb{R} \to \mathbb{R}$. For any site $z \in \mathbb{Z}^\nu$ define $V_z : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ by setting $V_z(\mathrm{x}) = V(q_z)$. We consider finite volume anharmonic Hamiltonians $H^{\Lambda_L} : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ of the form

$$H^{\Lambda_L} = H_h^{\Lambda_L} + \sum_{z \in \Lambda_L} V_z. \tag{1.8}$$

In order to prove Theorem 2, we need the following assumptions on V: $V \in C^2(\mathbb{R})$, $V' \in L^1(\mathbb{R})$, $V'' \in L^\infty(\mathbb{R})$, and

$$\kappa_V = \int |r| \left| \widehat{V'}(r) \right| dr < \infty.$$

Here $\widehat{V'}$ is the Fourier transform of V'. Under these assumptions, we prove a locality result analogous to Theorem 1. A specific class of observables, the Weyl functions, are particularly well-suited for our considerations, see Sect. 2.3 and Sect. 4. They are defined as follows. For any function $f : \Lambda_L \to \mathbb{C}$, the Weyl function generated by f, denoted by $W(f)$, is the observable $W(f) : \mathcal{X}_{\Lambda_L} \to \mathbb{C}$ given by

$$[W(f)](\mathrm{x}) = \exp\left[i \sum_{x \in \Lambda_L} \mathrm{Re}[f(x)] q_x + \mathrm{Im}[f(x)] p_x \right]. \tag{1.9}$$

Clearly, if f is supported in $X \subset \Lambda_L$, then $W(f)$ is supported in X as well. Moreover, it is easy to see that for any function $f : \Lambda_L \to \mathbb{C}$, $\|W(f)\|_\infty = 1$. Our next result is

Theorem 2 *Let $V : \mathbb{R} \to \mathbb{R}$ satisfy $V \in C^2(\mathbb{R})$, $V' \in L^1(\mathbb{R})$, $V'' \in L^\infty(\mathbb{R})$, and κ_V, as defined above, is finite. Take X and Y to be finite subsets of $\mathbb{Z}^\nu$ and let L_0 be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$, denote by α_t^L the dynamics corresponding to H^{Λ_L}. For any $\mu > 0$ and any functions $f, g : \Lambda_{L_0} \to \mathbb{C}$ with support of f in X and support of g in Y, there exist a positive numbers C and v_{ah}, both independent of L, such that the bound*

$$\left\| \left\{ \alpha_t^L(W(f)), W(g) \right\} \right\|_\infty \leq C \|f\|_\infty \|g\|_\infty \min(|X|, |Y|) e^{-\mu(d(X,Y) - v_{ah}|t|)} \tag{1.10}$$

holds for all $t \in \mathbb{R}$.

Through our method of proof, we are able to estimate the anharmonic Lieb-Robinson velocity, v_{ah}, appearing in the statement of Theorem 2 above. In fact, for any $\mu > 0$ and each $\epsilon > 0$,

$$v_{ah}(\mu) \leq \left(1 + \frac{\epsilon}{\mu}\right) v_h(\mu + \epsilon) + \frac{C\kappa_V}{\mu}, \tag{1.11}$$

and hence the anharmonic velocity can be bounded by an explicit perturbation of the harmonic velocity. For more details, see Sect. 3.

Let us briefly compare Theorem 2 with the results of [15] and [4]. In [15], two distinct anharmonic models are considered. The first is a multi-dimensional rotator model with a compact configuration space. For such a system, the existence of a finite Lieb-Robinson velocity is established, yet no explicit estimates are provided. Next, the authors consider a model with a quartic on-site term, i.e. they take $V_z(\mathrm{x}) = q_z^4$ as above. Beginning in [15] and later improved in [4], an important quasi-locality result is proven. More specifically, it is demonstrated that for almost every initial condition, measured with respect to a state satisfying a super-stability estimate, the relevant Poisson bracket is exponentially small in time whenever the distance between the supports of the local observables grows faster than $t \log^\alpha(t)$ for suitable $\alpha > 0$. Our methods do not apply to such a strong perturbation, however, the established result is insufficient to conclude the existence of a finite Lieb-Robinson velocity.

Using distinct perturbative methods, our results provide explicit estimates on the Lieb-Robinson velocity for a large class of anharmonicities. We do not assume compactness of configuration space nor do we make reference to a state satisfying a super-stability estimate. As we prove in Sect. 4, our methods also apply to multi-site perturbations with sufficiently fast decay.

The paper is organized as follows. In Sect. 2, we discuss our results concerning the Harmonic Hamiltonian and prove Theorem 1. Using an interpolation argument, we prove Theorem 2 in Sect. 3. This result demonstrates that the anharmonic velocity can be estimated in terms of the harmonic velocity and an additive shift which is quantifiable in terms of the perturbation. In Sect. 4, we generalize Theorem 2 to cover a wide class of multi-site perturbations. Finally, Sect. 5 contains a variety of useful solution estimates used throughout the paper. These well-known results we include for the sake of completeness.

2 The Harmonic Hamiltonian

The main goal of this section is to prove Theorem 1. For the convenience of the reader, we begin with a subsection describing some basic features of the harmonic Hamiltonian.

In particular, we reintroduce the Hamiltonian and find an explicit expression for the corresponding flow. In the subsections which follow, we prove two locality estimates, both with finite group velocities. The first is valid for a general class of smooth and bounded observables. The next holds for a special class of observables, the Weyl functions. This latter result will be particularly useful in subsequent sections.

2.1 Some Basics

For any integer $L \geq 1$, we consider subsets $\Lambda_L = (-L, L]^\nu \subset \mathbb{Z}^\nu$ and the finite volume harmonic Hamiltonian $H_h^{\Lambda_L} : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ given by

$$H_h^{\Lambda_L}(\mathrm{x}) = \sum_{x \in \Lambda_L} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{\nu} \lambda_j (q_x - q_{x+e_j})^2. \tag{2.1}$$

Here, for each $j = 1, \ldots, \nu$, the e_j are the canonical basis vectors in $\mathbb{Z}^\nu$, $\omega \geq 0$, and $\lambda_j \geq 0$. The model in (2.1) is defined with periodic boundary conditions, in the sense that $q_{x+e_j} = q_{x-(2L-1)e_j}$ if $x \in \Lambda_L$ but $x + e_j \notin \Lambda_L$.

Our first task is to provide an explicit expression for the flow corresponding to (2.1). In doing so, we will fix an integer value of $L \geq 1$ and drop its dependence in a variety of quantities to ease notation. Given any $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$ and $t \in \mathbb{R}$, the components of $\Phi_t^h(\mathrm{x}) = \{(q_x(t), p_x(t))\}_{x \in \Lambda_L}$ satisfy the following coupled system of differential equations: for each $x \in \Lambda_L$ and $t \in \mathbb{R}$,

$$\begin{aligned} \dot{q}_x(t) &= 2p_x(t), \\ \dot{p}_x(t) &= -2\omega^2 q_x(t) - 2\sum_{j=1}^{\nu} \lambda_j \left(2q_x(t) - q_{x+e_j}(t) - q_{x-e_j}(t)\right) \end{aligned} \tag{2.2}$$

with initial condition $\{(q_x(0), p_x(0))\}_{x \in \Lambda_L} = \mathrm{x}$. Introducing Fourier variables, the system defined by (2.2) decouples which leads to an exact solution. This is the content of Lemma 1 found below.

Before stating Lemma 1, it is useful to introduce some additional notation. Fourier sums will be defined via the set Λ_L^* given by

$$\Lambda_L^* = \left\{ \frac{x\pi}{L} : x \in \Lambda_L \right\}.$$

Note that $\Lambda_L^* \subset (-\pi, \pi]^\nu$ and $|\Lambda_L^*| = |\Lambda_L| = (2L)^\nu$. The following functions play an important role in our calculations. Suppose $\omega > 0$ and take $\gamma : \Lambda_L^* \to \mathbb{R}$ to be given by

$$\gamma(k) = \sqrt{\omega^2 + 4\sum_{j=1}^{\nu} \lambda_j \sin^2(k_j/2)}, \tag{2.3}$$

and for each $m \in \{-1, 0, 1\}$ and any $t \in \mathbb{R}$, set $h_t^{(m)} : \Lambda_L \to \mathbb{R}$ to be

$$h_t^{(-1)}(x) = \mathrm{Im}\left[\frac{1}{|\Lambda_L|} \sum_{k \in \Lambda_L^*} \frac{e^{i(k \cdot x - 2\gamma(k)t)}}{\gamma(k)} \right],$$

$$
\begin{aligned}
h_t^{(0)}(x) &= \mathrm{Re}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*} e^{i(k\cdot x-2\gamma(k)t)}\right], \\
h_t^{(1)}(x) &= \mathrm{Im}\left[\frac{1}{|\Lambda_L|}\sum_{k\in\Lambda_L^*} \gamma(k)\, e^{i(k\cdot x-2\gamma(k)t)}\right].
\end{aligned} \tag{2.4}
$$

Each of these functions depend on the length scale L, however, we are suppressing that dependence.

Lemma 1 *Suppose* $\omega > 0$. *For any* $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$ *and* $t \in \mathbb{R}$, *the mapping* $\Phi_t^h : \mathcal{X}_{\Lambda_L} \to \mathcal{X}_{\Lambda_L}$ *is well-defined. In particular, for each* $x \in \Lambda_L$ *and* $t \in \mathbb{R}$, *the components of* $\Phi_t^h(\mathrm{x}) = \{(q_x(t), p_x(t))\}_{x\in\Lambda_L}$ *are given by*

$$
q_x(t) = \sum_{y\in\Lambda_L} q_y(0)\, h_t^{(0)}(x-y) - p_y(0)\, h_t^{(-1)}(x-y) \tag{2.5}
$$

and

$$
p_x(t) = \sum_{y\in\Lambda_L} q_y(0)\, h_t^{(1)}(x-y) + p_y(0)\, h_t^{(0)}(x-y). \tag{2.6}
$$

Here, if necessary, the function values $h_t^{(m)}(x-y)$ *are defined by periodic extension, and we regard* $\mathrm{x} = \{(q_x(0), p_x(0))\}_{x\in\Lambda_L}$.

Proof Taking a second derivative of (2.2), we find that for each $x \in \Lambda_L$ and any $t \in \mathbb{R}$,

$$
\begin{aligned}
\ddot{q}_x(t) &= -4\omega^2 q_x(t) - 4\sum_{j=1}^{\nu}\lambda_j\left(2q_x(t) - q_{x+e_j}(t) - q_{x-e_j}(t)\right), \\
\ddot{p}_x(t) &= -4\omega^2 p_x(t) - 4\sum_{j=1}^{\nu}\lambda_j\left(2p_x(t) - p_{x+e_j}(t) - p_{x-e_j}(t)\right).
\end{aligned} \tag{2.7}
$$

For any $k \in \Lambda_L^*$ and $t \in \mathbb{R}$, set

$$
Q_k(t) = \frac{1}{\sqrt{|\Lambda_L|}}\sum_{x\in\Lambda_L} e^{-ik\cdot x} q_x(t) \quad \text{and} \quad P_k(t) = \frac{1}{\sqrt{|\Lambda_L|}}\sum_{x\in\Lambda_L} e^{-ik\cdot x} p_x(t). \tag{2.8}
$$

Inserting (2.7) into the second derivative of (2.8), we find an equivalent system of uncoupled differential equations. In fact, for each $k \in \Lambda_L^*$ and any $t \in \mathbb{R}$,

$$
\begin{aligned}
\ddot{Q}_k(t) &= -4\omega^2 Q_k(t) - 4\sum_{j=1}^{\nu}\lambda_j\left(2 - e^{ik_j} - e^{-ik_j}\right) Q_k(t) = -4\gamma(k)^2 Q_k(t), \\
\ddot{P}_k(t) &= -4\omega^2 P_k(t) - 4\sum_{j=1}^{\nu}\lambda_j\left(2 - e^{ik_j} - e^{-ik_j}\right) P_k(t) = -4\gamma(k)^2 P_k(t),
\end{aligned} \tag{2.9}
$$

where γ is as in (2.3). The solution of (2.9) is given by

$$\begin{aligned} Q_k(t) &= C_k e^{-2i\gamma(k)t} + \overline{C_{-k}} e^{2i\gamma(k)t}, \\ P_k(t) &= D_k e^{-2i\gamma(k)t} + \overline{D_{-k}} e^{2i\gamma(k)t}, \end{aligned} \tag{2.10}$$

where $-k$ is defined to be the element of Λ_L^* whose components are given by

$$(-k)_j = \begin{cases} -k_j, & \text{if } |k_j| < \pi, \\ \pi, & \text{otherwise.} \end{cases}$$

The relationship between the coefficients in (2.10) above is derived using the fact that the initial condition is real-valued, e.g.,

$$Q_k(0) = \overline{Q_{-k}(0)} \quad \text{and} \quad \dot{Q}_k(0) = \overline{\dot{Q}_{-k}(0)}.$$

Using Fourier inversion, we recover the components of the flow from (2.10). In fact,

$$\begin{aligned} q_x(t) &= \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k \in \Lambda_L^*} e^{ik\cdot x} Q_k(t) \\ &= \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k \in \Lambda_L^*} C_k e^{i(k\cdot x - 2\gamma(k)t)} + \overline{C_k} e^{-i(k\cdot x - 2\gamma(k)t)}, \end{aligned} \tag{2.11}$$

and similarly, we find that

$$p_x(t) = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k \in \Lambda_L^*} D_k e^{i(k\cdot x - 2\gamma(k)t)} + \overline{D_k} e^{-i(k\cdot x - 2\gamma(k)t)}. \tag{2.12}$$

To express these solutions explicitly in terms of the initial condition, we observe that

$$Q_k(0) = C_k + \overline{C_{-k}} \quad \text{and} \quad P_k(0) = D_k + \overline{D_{-k}}, \tag{2.13}$$

and introduce

$$B_k = \frac{1}{\sqrt{2\gamma(k)}} P_k(0) - i\sqrt{\frac{\gamma(k)}{2}} Q_k(0) \quad \text{with } \overline{B_k} = \frac{1}{\sqrt{2\gamma(k)}} P_{-k}(0) + i\sqrt{\frac{\gamma(k)}{2}} Q_{-k}(0). \tag{2.14}$$

It is easy to see that

$$Q_k(0) = \frac{i}{\sqrt{2\gamma(k)}} \left(B_k - \overline{B_{-k}}\right) \quad \text{and} \quad P_k(0) = \sqrt{\frac{\gamma(k)}{2}} \left(B_k + \overline{B_{-k}}\right), \tag{2.15}$$

and therefore,

$$C_k = \frac{iB_k}{\sqrt{2\gamma(k)}} \quad \text{and} \quad D_k = \sqrt{\frac{\gamma(k)}{2}} B_k. \tag{2.16}$$

Plugging this into (2.11), we find that

$$
\begin{aligned}
q_x(t) &= \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \frac{iB_k}{\sqrt{2\gamma(k)}} e^{i(k\cdot x-2\gamma(k)t)} - \frac{i\overline{B_k}}{\sqrt{2\gamma(k)}} e^{-i(k\cdot x-2\gamma(k)t)} \\
&= \frac{1}{2\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} Q_k(0)e^{i(k\cdot x-2\gamma(k)t)} + \overline{Q_k(0)}e^{-i(k\cdot x-2\gamma(k)t)} \\
&\quad + \frac{i}{2\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \frac{P_k(0)}{\gamma(k)} e^{i(k\cdot x-2\gamma(k)t)} - \frac{\overline{P_k(0)}}{\gamma(k)} e^{-i(k\cdot x-2\gamma(k)t)} \\
&= \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \mathrm{Re}\left[Q_k(0)e^{i(k\cdot x-2\gamma(k)t)}\right] - \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k\in\Lambda_L^*} \mathrm{Im}\left[\frac{P_k(0)}{\gamma(k)} e^{i(k\cdot x-2\gamma(k)t)}\right].
\end{aligned}
\tag{2.17}
$$

Moreover, one finds that

$$
\mathrm{Re}\left[Q_k(0)e^{i(k\cdot x-2\gamma(k)t)}\right] = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{y\in\Lambda_L} q_y(0)\,\mathrm{Re}\left[e^{i(k\cdot(x-y)-2\gamma(k)t)}\right]
\tag{2.18}
$$

while

$$
\mathrm{Im}\left[\frac{P_k(0)}{\gamma(k)} e^{i(k\cdot x-2\gamma(k)t)}\right] = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{y\in\Lambda_L} p_y(0)\,\mathrm{Im}\left[\frac{1}{\gamma(k)} e^{i(k\cdot(x-y)-2\gamma(k)t)}\right].
\tag{2.19}
$$

With the functions $h_t^{(m)}$, as defined in (2.4), we conclude that

$$
q_x(t) = \sum_{y\in\Lambda_L} q_y(0)h_t^{(0)}(x-y) - p_y(0)h_t^{(-1)}(x-y),
\tag{2.20}
$$

as claimed in (2.5). A similar calculation yields (2.6). Since the functions $h_t^{(m)}$ are real valued, so too are the solutions $q_x(t)$ and $p_x(t)$. This proves Lemma 1. □

Remark 1 An analogue of (2.5) and (2.6) holds in the event that $\omega = 0$. This is seen by proceeding as in the proof of Lemma 1 and observing that now $\gamma(0) = 0$, but $\gamma(k) \neq 0$ for $k \neq 0$. For $k \neq 0$, the formulas above are correct, and a simple calculation shows that, in this case,

$$
\begin{aligned}
Q_0(t) &= Q_0(0) + 2P_0(0)t, \\
P_0(t) &= P_0(0),
\end{aligned}
\tag{2.21}
$$

similar to (2.10). One easily sees that (2.5) and (2.6) still hold with the convention that

$$
h_t^{(-1)}(x) = -\frac{2t}{|\Lambda_L|} + \mathrm{Im}\left[\frac{1}{|\Lambda_L|} \sum_{k\in\Lambda_L^*\setminus\{0\}} \frac{e^{i(k\cdot x-2\gamma(k)t)}}{\gamma(k)}\right].
\tag{2.22}
$$

We end this subsection with the following crucial estimate which was proven in [11].

Lemma 2 *Fix $L \geq 1$ and consider the functions $h_t^{(m)}$ as defined in (2.4) for $m \in \{-1, 0, 1\}$. For any $\mu > 0$, the bounds*

$$\begin{aligned} \left| h_t^{(0)}(x) \right| &\leq e^{-\mu(|x| - c_{\omega,\lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)}, \\ \left| h_t^{(-1)}(x) \right| &\leq c_{\omega,\lambda}^{-1} e^{-\mu(|x| - c_{\omega,\lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)}, \\ \left| h_t^{(1)}(x) \right| &\leq c_{\omega,\lambda} e^{\mu/2} e^{-\mu(|x| - c_{\omega,\lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \end{aligned} \tag{2.23}$$

hold for all $t \in \mathbb{R}$ and $x \in \Lambda_L$. Here $|x| = \sum_{j=1}^{\nu} |x_i|$ and one may take $c_{\omega,\lambda} = (\omega^2 + 4\sum_{j=1}^{\nu} \lambda_j)^{1/2}$.

We refer the interested reader to Lemma 3.7 of [11] for the proof. Moreover, we stress that Lemma 2 is valid for all $\omega \geq 0$.

2.2 A General Locality Estimate

Our first locality bound for the harmonic Hamiltonian follows directly from Lemmas 1 and 2. We state this as Theorem 3. In fact, it was observed in [15] that, in general, the Poisson brackets used to describe locality of the dynamics can be bounded by partial derivatives of the corresponding solutions of Hamilton's equations with respect to the initial conditions. Theorem 3 below follows from this observation, the explicit form of the harmonic solutions we demonstrated in Lemma 1, and the estimates we proved in Lemma 2. As we will see, Theorem 1 is an immediate consequence of Theorem 3. Recall that we have defined $\mathcal{A}_{\Lambda_L}^{(1)}$ to be the set of observables $A \in \mathcal{A}_{\Lambda_L}$ for which: given any $x \in \Lambda_L$, $\frac{\partial A}{\partial q_x} \in \mathcal{A}_{\Lambda_L}$, $\frac{\partial A}{\partial p_x} \in \mathcal{A}_{\Lambda_L}$, and

$$\|\partial A\|_\infty = \sup_{x \in \Lambda_L} \max \left(\left\| \frac{\partial A}{\partial q_x} \right\|_\infty, \left\| \frac{\partial A}{\partial p_x} \right\|_\infty \right) < \infty. \tag{2.24}$$

Theorem 3 *Let X and Y be finite subsets of $\mathbb{Z}^\nu$ and take L_0 to be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$, let $\alpha_t^{h,L}$ denote the dynamics corresponding to $H_h^{\Lambda_L}$. For any $\mu > 0$ and any observables $A, B \in \mathcal{A}_{\Lambda_{L_0}}^{(1)}$ with support of A in X and support of B in Y, the bound*

$$\left\| \left\{ \alpha_t^{h,L}(A), B \right\} \right\|_\infty \leq C \|\partial A\|_\infty \|\partial B\|_\infty \sum_{x \in X, y \in Y} e^{-\mu(d(x,y) - c_{\omega,\lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \tag{2.25}$$

holds for all $t \in \mathbb{R}$. Here

$$d(x, y) = \sum_{j=1}^{\nu} \min_{\eta_j \in \mathbb{Z}} |x_j - y_j + 2L\,\eta_j| \tag{2.26}$$

is the distance on the torus and the constants may be taken as $C = (2 + c_{\omega,\lambda} e^{\mu/2} + c_{\omega,\lambda}^{-1})$ with $c_{\omega,\lambda} = (\omega^2 + 4\sum_{j=1}^{\nu} \lambda_j)^{1/2}$.

Proof The Poisson bracket is easy to calculate. In fact, for any $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$,

$$\left[\left\{\alpha_t^{h,L}(A), B\right\}\right](\mathrm{x}) = \sum_{y \in Y} \frac{\partial}{\partial q_y} A\left(\Phi_t^{h,L}(\mathrm{x})\right) \cdot \frac{\partial B}{\partial p_y}(\mathrm{x}) - \frac{\partial}{\partial p_y} A\left(\Phi_t^{h,L}(\mathrm{x})\right) \cdot \frac{\partial B}{\partial q_y}(\mathrm{x}). \quad (2.27)$$

By the chain rule,

$$\frac{\partial}{\partial q_y} A\left(\Phi_t^{h,L}(\mathrm{x})\right) = \sum_{x \in X} \frac{\partial A}{\partial q_x}\left(\Phi_t^{h,L}(\mathrm{x})\right) \cdot \frac{\partial q_x}{\partial q_y}(t) + \frac{\partial A}{\partial p_x}\left(\Phi_t^{h,L}(\mathrm{x})\right) \cdot \frac{\partial p_x}{\partial q_y}(t) \quad (2.28)$$

and a similar formula holds for $\frac{\partial}{\partial p_y} A(\Phi_t^{h,L}(\mathrm{x}))$. Now estimating (2.27), we find that

$$\begin{aligned} \left\|\left\{\alpha_t^{h,L}(A), B\right\}\right\|_\infty \leq \|\partial A\|_\infty \|\partial B\|_\infty \sum_{x \in X, y \in Y} & \left|h_t^{(-1)}(x-y)\right| + 2\left|h_t^{(0)}(x-y)\right| \\ & + \left|h_t^{(1)}(x-y)\right|, \end{aligned} \quad (2.29)$$

using Lemma 1. The bound in (2.25) now follows from Lemma 2. □

From Theorem 3, and specifically the bound (2.25), we see that for any $\mu > 0$, the harmonic velocity v_h is essentially described by

$$v_h(\mu) = c_{\omega,\lambda} \max\left(\frac{2}{\mu}, e^{(\mu/2)+1}\right). \quad (2.30)$$

In fact, given (2.25) for some $\mu > 0$, it is easy to see that for any $0 < \epsilon < 1$,

$$\sum_{x \in X, y \in Y} e^{-\mu d(x,y)} \leq e^{-\epsilon \mu d(X,Y)} \min(|X|, |Y|) \sum_{z \in \Lambda_L} e^{-\mu(1-\epsilon) d(0,z)}, \quad (2.31)$$

where we have set $d(X,Y) = \min_{x \in X, y \in Y} d(x,y)$. Thus, Theorem 1 is a simple consequence of Theorem 3. It is interesting to note that for any L the quantity

$$\sum_{z \in \Lambda_L} e^{-\mu(1-\epsilon) d(0,z)} \leq \sum_{z \in \mathbb{Z}^\nu} e^{-\mu(1-\epsilon)|z|}, \quad (2.32)$$

where $|z|$ denotes the L^1-metric on $\mathbb{Z}^\nu$. Given this and the fact that, for sufficiently large L, the distance $d(X,Y)$ agrees with the L^1-distance between X and Y, it is clear that the estimate proven in Theorem 1 is genuinely independent of the length scale L.

Since the bounds are valid for any $\mu > 0$, Theorem 3 demonstrates arbitrarily fast exponential decay in space with a velocity that depends on μ. Typically, however, one is interested in the best possible estimates on v_h given some decay rate. In this sense, the optimal harmonic velocity, as described by (2.30), occurs when the equation

$$\frac{\mu}{2} = e^{(\mu/2)+1} \quad (2.33)$$

holds. It is easy to see that the solution to (2.33), denoted by μ_0, satisfies $1/2 < \mu_0 < 1$, and therefore the corresponding velocity $v_h(\mu_0) \leq 4c_{\omega,\lambda}$.

2.3 The Harmonic Evolution of Weyl Functions

In preparation for our arguments in Sects. 3 and 4, we will now present a different proof of our locality result, analogous to Theorem 3, valid for Weyl functions. Recall that a Weyl function is an observable, generated by a function $f : \Lambda_L \to \mathbb{C}$, with the form

$$[W(f)](\mathrm{x}) = \exp\left[i \sum_{x \in \Lambda_L} \mathrm{Re}[f(x)] q_x + \mathrm{Im}[f(x)] p_x \right]. \tag{2.34}$$

One important property of the Weyl functions is typically referred to as the Weyl relation. We state this as Proposition 4.

Proposition 4 (Weyl Relation) *Let $f, g : \Lambda_L \to \mathbb{C}$. We have that*

$$\{W(f), W(g)\} = -\mathrm{Im}[\langle f, g\rangle] W(f) W(g), \tag{2.35}$$

where the inner product is taken in $\ell^2(\Lambda_L)$.

Proof A direct calculation yields

$$\begin{aligned}\{W(f), W(g)\} &= \sum_{x \in \Lambda_L} \frac{\partial}{\partial q_x} W(f) \frac{\partial}{\partial p_x} W(g) - \frac{\partial}{\partial p_x} W(f) \frac{\partial}{\partial q_x} W(g) \\ &= \sum_{x \in \Lambda_L} (-\mathrm{Re}[f(x)]\,\mathrm{Im}[g(x)] + \mathrm{Im}[f(x)]\,\mathrm{Re}[g(x)]) W(f) W(g).\end{aligned}$$

Noting that

$$\begin{aligned}\mathrm{Im}[\langle f, g\rangle] &= \mathrm{Im}\left[\sum_{x \in \Lambda_L} \overline{f(x)} g(x) \right] \\ &= \sum_{x \in \Lambda_L} (-\mathrm{Im}[f(x)]\,\mathrm{Re}[g(x)] + \mathrm{Re}[f(x)]\,\mathrm{Im}[g(x)])\end{aligned}$$

proves the proposition. □

Another useful property of the Weyl functions is that the harmonic dynamics leaves this class of observables invariant. This important fact, which follows immediately from Lemma 1, is the content of the next proposition. Before stating this, it is convenient to introduce notation for the convolution of two functions $f, g : \Lambda_L \to \mathbb{C}$,

$$(f * g)(x) = \sum_{y \in \Lambda_L} f(y) g(x - y), \tag{2.36}$$

where, if necessary, $g(x - y)$ is calculated by periodic extension.

Proposition 5 *Let $f : \Lambda_L \to \mathbb{C}$ and take $t \in \mathbb{R}$.*

$$\alpha_t^{h,L}(W(f)) = W(f_t), \tag{2.37}$$

where

$$f_t = f * \overline{\left(h_t^{(0)} + \frac{i}{2}(h_t^{(-1)} + h_t^{(1)})\right)} + \overline{f} * \left(\frac{i}{2}(h_t^{(1)} - h_t^{(-1)})\right) \tag{2.38}$$

with $h_t^{(-1)}$, $h_t^{(0)}$, *and* $h_t^{(1)}$ *as in* (2.4).

Proof For any point $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$, we have that

$$\begin{aligned}
&\left[\alpha_t^{h,L}(W(f))\right](\mathrm{x}) \\
&= \exp\left(i \sum_{x\in\Lambda_L} \mathrm{Re}[f(x)]q_x(t) + \mathrm{Im}[f(x)]p_x(t)\right) \\
&= \exp\left(i \sum_{x\in\Lambda_L} \mathrm{Re}[f(x)] \sum_{y\in\Lambda_L} q_y(0)\, h_t^{(0)}(x-y) - p_y(0)\, h_t^{(-1)}(x-y)\right) \\
&\quad\times \exp\left(i \sum_{x\in\Lambda_L} \mathrm{Im}[f(x)] \sum_{y\in\Lambda_L} q_y(0)\, h_t^{(1)}(x-y) + p_y(0)\, h_t^{(0)}(x-y)\right) \\
&= \exp\left(i \sum_{y\in\Lambda_L} q_y(0) \sum_{x\in\Lambda_L} \mathrm{Re}[f(x)]h_t^{(0)}(x-y) + \mathrm{Im}[f(x)]\, h_t^{(1)}(x-y)\right) \\
&\quad\times \exp\left(i \sum_{y\in\Lambda_L} p_y(0) \sum_{x\in\Lambda_L} \mathrm{Im}[f(x)]\, h_t^{(0)}(x-y) - \mathrm{Re}[f(x)]h_t^{(-1)}(x-y)\right) \\
&= \left[W(f_t)\right](\mathrm{x}),
\end{aligned} \tag{2.39}$$

where we have defined the function $f_t : \Lambda_L \to \mathbb{C}$ by (2.38). □

It is obvious that Theorem 6 below follows immediately from Theorem 3, since the Weyl functions are clearly in $\mathcal{A}_{\Lambda_L}^{(1)}$. We will here give a different, but equally short, proof which uses the specific properties of Weyl functions.

Theorem 6 *Let X and Y be finite subsets of $\mathbb{Z}^\nu$ and take L_0 to be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $\mu > 0$, $L \geq L_0$, and any functions $f, g : \Lambda_{L_0} \to \mathbb{C}$ with support of f in X and support of g in Y, the bound*

$$\left\| \left\{\alpha_t^{h,L}(W(f)), W(g)\right\} \right\|_\infty \leq C \|f\|_\infty \|g\|_\infty \sum_{x\in X, y\in Y} e^{-\mu(d(x,y) - c_{\omega,\lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \tag{2.40}$$

holds for all $t \in \mathbb{R}$. Here, as in (2.26), $d(x, y)$ is the distance on the torus and the constants may be taken as $C = (1 + c_{\omega,\lambda} e^{\mu/2} + c_{\omega,\lambda}^{-1})$ with $c_{\omega,\lambda} = (\omega^2 + 4\sum_{j=1}^{\nu} \lambda_j)^{1/2}$.

Proof Combining Propositions 5 and 4, it is clear that

$$\left\{\alpha_t^{h,L}(W(f)), W(g)\right\} = \{W(f_t), W(g)\} = -\,\mathrm{Im}\left[\langle f_t, g\rangle\right] W(f_t)W(g). \tag{2.41}$$

In this case, the bound

$$\left\| \left\{ \alpha_t^{h,L}(W(f)), W(g) \right\} \right\|_\infty \le \left| \mathrm{Im}\left[\langle f_t, g \rangle \right] \right|, \tag{2.42}$$

readily follows. Appealing again to Proposition 5, we have that for any $y \in \Lambda_L$,

$$\begin{aligned} f_t(y) = & \sum_{x \in X} f(x) \left(h_t^{(0)}(x-y) - \frac{i}{2} h_t^{(-1)}(x-y) - \frac{i}{2} h_t^{(1)}(x-y) \right) \\ & + \sum_{x \in X} \overline{f(x)} \left(\frac{i}{2} h_t^{(1)}(x-y) - \frac{i}{2} h_t^{(-1)}(x-y) \right), \end{aligned} \tag{2.43}$$

and therefore,

$$\begin{aligned} |\langle f_t, g \rangle| &= \left| \sum_{y \in Y} \overline{f_t(y)} g(y) \right| \\ &\le \|f\|_\infty \|g\|_\infty \sum_{x \in X, y \in Y} |h_t^{(0)}(x-y)| + |h_t^{(-1)}(x-y)| + |h_t^{(1)}(x-y)|. \end{aligned} \tag{2.44}$$

Theorem 6 now follows from Lemma 2. □

We end this section with a corollary of Theorem 6 that will be particularly useful in the next sections. The locality bound we prove for the anharmonic dynamics is derived by iterating a certain inequality involving the harmonic estimate. With this in mind, it is useful to introduce the following family of decaying functions. For any $\mu > 0$, consider $F_\mu : [0, \infty) \to (0, \infty)$ defined by

$$F_\mu(r) = \frac{e^{-\mu r}}{(1+r)^{\nu+1}}. \tag{2.45}$$

Clearly, these function F_μ also depend on the quantity $\nu \ge 1$, which is the dimension of the underlying lattice in our models, but we will suppress that dependence in our notation. Unlike the bare exponential $e^{-\mu r}$, these functions have the following nice property. There exists a number $C_\nu > 0$ for which, given any pair of sites $x, y \in \mathbb{Z}^\nu$,

$$\sum_{z \in \mathbb{Z}^\nu} F_\mu(|x-z|) F_\mu(|z-y|) \le C_\nu F_\mu(|x-y|). \tag{2.46}$$

Here one may take

$$C_\nu = 2^{\nu+1} \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|z|)^{\nu+1}}. \tag{2.47}$$

Functions of this type were introduced in [10], see also [11], as an aide in proving Lieb-Robinson bounds. We will use them here as well.

We can rewrite the decay expressed in our harmonic estimates, i.e. (2.25), in terms of these functions F_μ.

Corollary 1 *Let X and Y be finite subsets of $\mathbb{Z}^\nu$ and take L_0 to be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $\mu > 0$, $\epsilon > 0$, $L \ge L_0$, and any functions $f, g : \Lambda_{L_0} \to \mathbb{C}$ with* *support of f in X and support of g in Y, the bound*

$$\left\| \left\{ \alpha_t^{h,L}(W(f)), W(g) \right\} \right\|_\infty \leq C \, \|f\|_\infty \, \|g\|_\infty \, e^{(\mu+\epsilon)v_h(\mu+\epsilon)\,|t|} \sum_{x \in X, y \in Y} F_\mu(d(x,y)), \tag{2.48}$$

holds for all $t \in \mathbb{R}$. Here

$$C = (1 + c_{\omega,\lambda} e^{\frac{(\mu+\epsilon)}{2}} + c_{\omega,\lambda}^{-1}) \sup_{s \geq 0} \left[(1+s)^{\nu+1} e^{-\epsilon s} \right] \tag{2.49}$$

and v_h is as defined in (2.30).

3 On-Site Anharmonicities

In this section, we will prove a locality result, analogous to Theorem 6, for a specific class of perturbations of the harmonic Hamiltonian. A much more general result, which follows from the same basic arguments, is presented in the next section. We begin with a precise statement of the models we consider, and then prove the result.

To make our basic technique more transparent, we will only consider on-site potentials that are generated by a particular function V in this section, see Sect. 4 for a more general result. Let $V : \mathbb{R} \to \mathbb{R}$ satisfy $V \in C^2(\mathbb{R})$, $V' \in L^1(\mathbb{R})$, $V'' \in L^\infty(\mathbb{R})$, and suppose further that the quantity

$$\kappa_V = \int_{\mathbb{R}} |r| \, |\widehat{V'}(r)| \, dr \tag{3.1}$$

is finite. Here $\widehat{V'}$ is the Fourier transform of V'. Given such a function V and an integer $L \geq 1$, we define an anharmonic Hamiltonian $H^{\Lambda_L} : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ by setting

$$H^{\Lambda_L} = H_h^{\Lambda_L} + \sum_{z \in \Lambda_L} V_z, \tag{3.2}$$

where for each $z \in \Lambda_L$, the function $V_z : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ is given by $V_z(\mathrm{x}) = V(q_z)$.

As is discussed at the end of Sect. 2.3, we will state our result in terms of the functions $F_\mu : [0, \infty) \to (0, \infty)$ given by

$$F_\mu(r) = \frac{e^{-\mu r}}{(1+r)^{\nu+1}}, \tag{3.3}$$

with $\nu > 0$ corresponding to the dimension of the underlying lattice $\mathbb{Z}^\nu$. The goal of this section is to prove the following result.

Theorem 7 *Suppose $V : \mathbb{R} \to \mathbb{R}$ satisfies $V \in C^2(\mathbb{R})$, $V' \in L^1(\mathbb{R})$, $V'' \in L^\infty(\mathbb{R})$, and κ_V, as in* (3.1) *above, is finite. Let X and Y be finite subsets of $\mathbb{Z}^\nu$ and take L_0 to be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$ and $t \in \mathbb{R}$, let α_t^L denote the dynamics corresponding to H^{Λ_L}. Then, for any $\mu > 0$, $\epsilon > 0$, and any functions $f, g : \Lambda_{L_0} \to \mathbb{C}$ with support of f in X and support of g in Y, the bound*

$$\left\| \left\{ \alpha_t^L(W(f)), W(g) \right\} \right\|_\infty \leq C \, \|f\|_\infty \|g\|_\infty \, e^{\delta|t|} \sum_{x \in X, y \in Y} F_\mu\left(d(x,y)\right) \tag{3.4}$$

holds for all $t \in \mathbb{R}$. Here one may take

$$C = (1 + c_{\omega,\lambda} e^{\frac{(\mu+\epsilon)}{2}} + c_{\omega,\lambda}^{-1}) \sup_{s \geq 0} \left[(1+s)^{\nu+1} e^{-\epsilon s} \right] \tag{3.5}$$

and

$$\delta = \delta(\mu, \epsilon) = (\mu + \epsilon) v_h(\mu + \epsilon) + C C_\nu \kappa_V \tag{3.6}$$

where v_h is as in (2.30), *C_ν is in* (2.47), *and κ_V is in* (3.1).

Before we prove Theorem 7, we comment on the corresponding anharmonic velocity. With arguments similar to those given after the proof of Theorem 3, it is clear that Theorem 7 implies Theorem 2. In this case, we find that an upper bound on the anharmonic velocity for this model is

$$v_{ah}(\mu, \epsilon) = \left(1 + \frac{\epsilon}{\mu} \right) v_h(\mu + \epsilon) + \frac{C C_\nu \kappa_V}{\mu}. \tag{3.7}$$

We now present the proof.

Proof of Theorem 7 Our proof of this estimate is perturbative, and we begin by interpolating between the harmonic and anharmonic dynamics. Fix $L \geq L_0$ as in the statement of the theorem. Since we will regard both the harmonic and anharmonic dynamics on the same volume Λ_L, we drop the dependence of each on L. Observe that for any $t > 0$,

$$\{\alpha_t(W(f)), W(g)\} - \left\{ \alpha_t^h(W(f)), W(g) \right\} = \int_0^t \frac{d}{ds} \left\{ \alpha_s\left(\alpha_{t-s}^h(W(f)) \right), W(g) \right\} ds. \tag{3.8}$$

Moreover, a direct calculation shows that

$$\begin{aligned} \frac{d}{ds} \alpha_s\left(\alpha_{t-s}^h(W(f)) \right) &= \alpha_s\left(\left\{ \alpha_{t-s}^h(W(f)), H \right\} \right) - \alpha_s\left(\alpha_{t-s}^h(\{W(f), H_h\}) \right) \\ &= \alpha_s\left(\left\{ \alpha_{t-s}^h(W(f)), H - H_h \right\} \right) \\ &= \sum_{z \in \Lambda_L} \alpha_s\left(\left\{ \alpha_{t-s}^h(W(f)), V_z \right\} \right). \end{aligned} \tag{3.9}$$

The Poisson bracket on the right-hand side of (3.9) can be simplified

$$\left\{ \alpha_{t-s}^h(W(f)), V_z \right\} = \{W(f_{t-s}), V_z\} = -i \operatorname{Im}\left[f_{t-s}(z) \right] W(f_{t-s}) V_z'. \tag{3.10}$$

For the first equality above we used Proposition 5, and we have denoted by V_z' the function $V_z' : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ with $V_z'(\mathrm{x}) = V'(q_z)$.

These calculations lead to a particularly simple differential equation and thus, eventually, the bound (3.17) appearing below. In fact, for $t > 0$ fixed and $0 \leq s \leq t$, define the function

$$\Psi_t(s) = \{\alpha_s(\alpha_{t-s}^h(W(f))), W(g)\}. \tag{3.11}$$

We have shown that

$$\frac{d}{ds}\Psi_t(s) = \sum_{z\in\Lambda_L} \left\{\alpha_s\left(\left\{\alpha^h_{t-s}(W(f)), V_z\right\}\right), W(g)\right\}$$
$$= i\mathcal{L}_t(s)\Psi_t(s) + \mathcal{Q}_t(s), \tag{3.12}$$

where

$$\mathcal{L}_t(s) = -\sum_{z\in\Lambda_L} \mathrm{Im}\left[f_{t-s}(z)\right]\alpha_s(V'_z),$$
$$\mathcal{Q}_t(s) = -i\sum_{z\in\Lambda_L} \mathrm{Im}\left[f_{t-s}(z)\right]\alpha_s\left(\alpha^h_{t-s}(W(f))\right)\left\{\alpha_s(V'_z), W(g)\right\}, \tag{3.13}$$

and the final equality in (3.12) follows from an application of the Leibnitz rule for Poisson brackets. Since for each fixed s, $\mathcal{L}_t(s)$ is a real-valued function of phase space, the solution U_t of

$$\frac{d}{ds}U_t(s) = -i\mathcal{L}_t(s)U_t(s) \quad \text{with } U_t(0) = 1, \tag{3.14}$$

is a complex exponential. In addition, it is easy to see that

$$\frac{d}{ds}\left(\Psi_t(s)U_t(s)\right) = \mathcal{Q}_t(s)U_t(s), \tag{3.15}$$

and therefore,

$$\Psi_t(t)U_t(t) = \Psi_t(0) + \int_0^t \mathcal{Q}_t(s)U_t(s)\,ds, \tag{3.16}$$

from which

$$\|\{\alpha_t(W(f)), W(g)\}\|_\infty \leq \left\|\left\{\alpha^h_t(W(f)), W(g)\right\}\right\|_\infty$$
$$+ \sum_{z\in\Lambda_L}\int_0^t \left|\mathrm{Im}\left[f_{t-s}(z)\right]\right| \left\|\left\{\alpha_s(V'_z), W(g)\right\}\right\|_\infty ds, \tag{3.17}$$

readily follows.

Now, if V'_z was a Weyl function, then we could immediately iterate the inequality in (3.17) and derive a bound. This is not the case, however, our assumptions on V allow us to write V'_z as an average of Weyl functions through its Fourier representation. In fact, we write the Fourier transform of V' as

$$\widehat{V'}(r) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iqr}\, V'(q)\,dq, \tag{3.18}$$

and by inversion, one has that

$$V'(q) = \int_{\mathbb{R}} e^{irq}\,\widehat{V'}(r)\,dr. \tag{3.19}$$

This implies that, as a function of phase space, V'_z can be expressed as

$$V'_z = \int_{\mathbb{R}} W(r\delta_z)\widehat{V'}(r)\,dr \tag{3.20}$$

where $r\delta_z : \Lambda_L \to \mathbb{R}$ is the function that has value r at z and 0 otherwise. Inserting (3.20) into (3.17), we have that

$$
\begin{aligned}
&\|\{\alpha_t(W(f)), W(g)\}\|_\infty \\
&\quad \leq \left\| \left\{ \alpha_t^h(W(f)), W(g) \right\} \right\|_\infty \\
&\qquad + \sum_{z \in \Lambda_L} \int_0^t \left| \mathrm{Im}\left[f_{t-s}(z) \right] \right| \int_{\mathbb{R}} \left| \widehat{V}'(r) \right| \left\| \left\{ \alpha_s\left(W(r\delta_z) \right), W(g) \right\} \right\|_\infty dr\, ds. \qquad (3.21)
\end{aligned}
$$

At this stage, we can finally iterate the inequality. First, however, we insert the harmonic bound found in Corollary 1.

Recall that for any $\mu > 0$ and $\epsilon > 0$ we have established (2.48) with a constant C as in (3.5). With equation (2.43), it is easy to see that, for any $\mu > 0$ and $\epsilon > 0$

$$
|\mathrm{Im}[f_t(z)]| \leq C \, \|f\|_\infty \, e^{(\mu+\epsilon)v_h(\mu+\epsilon)|t|} \sum_{x \in X} F_\mu(d(x,z)), \qquad (3.22)
$$

also holds for any $z \in \Lambda_L$ and $t \in \mathbb{R}$. To ease the notation a bit, we will denote by $\tilde{v} = (\mu+\epsilon)v_h(\mu+\epsilon)$. Using these bounds, the inequality in (3.21) now takes the form

$$
\begin{aligned}
&\|\{\alpha_t(W(f)), W(g)\}\|_\infty \\
&\quad \leq C \, \|f\|_\infty \, \|g\|_\infty \, e^{\tilde{v}t} \sum_{x \in X, y \in Y} F_\mu(d(x,y)) \\
&\qquad + C \, \|f\|_\infty \sum_{x \in X} \sum_{z \in \Lambda_L} F_\mu(d(x,z)) \int_0^t e^{\tilde{v}(t-s)} \int_{\mathbb{R}} |\widehat{V}'(r)| \left\| \left\{ \alpha_s\left(W(r\delta_z) \right), W(g) \right\} \right\|_\infty dr\, ds.
\end{aligned}
\qquad (3.23)
$$

Upon iterating (3.23) $m \geq 1$ times, we find that

$$
\|\{\alpha_t(W(f)), W(g)\}\|_\infty \leq C \, \|f\|_\infty \, \|g\|_\infty \, e^{\tilde{v}t} \sum_{x \in X, y \in Y} \sum_{n=0}^m a_n(x,y;t) + R_{m+1}(t), \qquad (3.24)
$$

where

$$
a_0(x,y;t) = F_\mu(d(x,y)), \qquad (3.25)
$$

$$
\begin{aligned}
a_1(x,y;t) &= Ct \int_{\mathbb{R}} |r| \left| \widehat{V}'(r) \right| dr \sum_{z \in \Lambda_L} F_\mu\left(d(x,z)\right) F_\mu\left(d(z,y)\right) \\
&\leq C \, \kappa_V \, C_\nu \, t F_\mu\left(d(x,y)\right), \qquad (3.26)
\end{aligned}
$$

and in general,

$$
\begin{aligned}
a_n(x,y;t) &= \frac{(Ct)^n}{n!} \left(\prod_{k=1}^n \int_{\mathbb{R}} |r_k| \left| \widehat{V}'(r_k) \right| dr_k \right) \sum_{z_1,\ldots,z_n \in \Lambda_L} F_\mu\left(d(x,z_1)\right) \cdots F_\mu\left(d(z_n,y)\right) \\
&\leq \frac{(C \, \kappa_V \, C_\nu \, t)^n}{n!} F_\mu\left(d(x,y)\right), \qquad (3.27)
\end{aligned}
$$

for any $1 \leq n \leq m$. In (3.26) and (3.27), we have used (2.46) several times.

From Lemma 6, found in Sect. 5, it is easy to see that the apriori estimate

$$\|\{\alpha_s(W(r\delta_z)), W(g)\}\|_\infty \leq C_1\,|Y|\,|r|\,\|g\|_\infty \exp\left(C_2\,t^2\right) \tag{3.28}$$

holds for all $0 \leq s \leq t$. Thus, for $t > 0$ fixed, the remainder term $R_{m+1}(t)$ converges to zero as $m \to \infty$. In fact,

$$R_{m+1}(t) \leq C_1\,|X|\,|Y|\,\|f\|_\infty \|g\|_\infty e^{\tilde{v}t + C_2 t^2} \frac{(C\,\kappa_V\,C_\nu t)^{m+1}}{(m+1)!}. \tag{3.29}$$

We have proven that

$$\|\{\alpha_t(W(f)), W(g)\}\|_\infty \leq C\,\|f\|_\infty\,\|g\|_\infty\,e^{(\tilde{v}+C\,\kappa_V\,C_\nu)t} \sum_{x\in X, y\in Y} F_\mu(d(x,y)), \tag{3.30}$$

i.e. (3.4) as claimed. □

4 Multiple Site Anharmonicities

In this section, we will generalize Theorem 7 in such a way that it covers perturbations involving long range interactions. As in the previous sections, we will be fixing some integer $L \geq 1$ and considering only finite volumes $\Lambda_L \subset \mathbb{Z}^\nu$.

We will introduce these perturbations quite generally and then discuss the assumptions necessary to prove our locality result. To each subset $Z \subset \Lambda_L$, we will assign a function $V(\cdot; Z) : \mathbb{R}^Z \to \mathbb{R}$ and a corresponding function of phase space $V_Z : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ defined by setting

$$V_Z(\mathrm{x}) = V\left(\{q_z\}_{z\in Z}; Z\right). \tag{4.1}$$

Here $\{q_z\}_{z\in Z}$ is regarded as a vector in $\mathbb{R}^Z$ and the number $V(\{q_z\}_{z\in Z}; Z)$ is calculated by evaluating $V(\cdot; Z)$ with q_z as the value in the z-th component for each $z \in Z$. With this understanding, we will use the notation

$$\partial_z V_Z = \frac{\partial}{\partial q_z} V_Z = \partial_z V(\cdot; Z), \tag{4.2}$$

to denote the partial derivatives of V_Z.

In general, the finite volume anharmonic Hamiltonians we consider are of the form $H^{\Lambda_L} : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$ with

$$H^{\Lambda_L} = H_h^{\Lambda_L} + \sum_{Z\subset\Lambda_L} V_Z, \tag{4.3}$$

where the sum above is over all subsets of Λ_L. As we saw in Sect. 3, in order to prove our locality result, we need some assumptions on the functions V_Z. We will now list these explicitly below.

First, we use Lemma 4, proven in Sect. 5, to provide explicit bounds on the components of the flow which, in particular, prevent the solutions from blowing-up in finite time. For these estimates, we assume the perturbation above satisfies:

(i) For each $Z \subset \Lambda_L$, the function V_Z has well-defined first order partial derivatives.
(ii) There exist numbers $C_1 \geq 0$, $\tilde{C}_1 \geq 0$, and $\mu_1 \geq 0$ such that for each $x \in \Lambda_L$ and any $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$,

$$\left(\sum_{Z \subset \Lambda_L} |\partial_x V_Z(\mathrm{x})| \right)^2 \leq C_1 \sum_{y \in \Lambda_L} (q_y^2 + \tilde{C}_1) F_{\mu_1}(d(x, y)) . \tag{4.4}$$

The decaying functions F_μ are as defined at the end of Sect. 2.3.

Next, much like in the proof of Theorem 7, we will need an apriori estimate on the Poisson bracket of specific, dynamically evolved observables. This is the content of Lemma 6 found in the next section. To prove it we use Lemma 5, and therefore, we must assume

(iii) For each $Z \subset \Lambda_L$, the function V_Z has well-defined second order partial derivatives.
(iv) There exist numbers $C_2 \geq 0$ and $\mu_2 \geq 0$ for which: given any pair $x, y \in \Lambda_L$, the bound

$$\sum_{Z \subset \Lambda_L} \left| \left[\partial_x \partial_y V_Z \right] (\mathrm{x}) \right| \leq C_2 F_{\mu_2}(d(x, y)) , \tag{4.5}$$

holds for all points $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$.

Lastly, we need the quantities that arise in our iteration scheme to be well-defined. For this we assume

(v) For each $Z \subset \Lambda_L$, the first order partial derivatives of V_Z are integrable. By this we mean that given $Z \subset \Lambda_L$ and $z \in Z$, the function $\partial_z V(\cdot; Z)$ is in $L^1(\mathbb{R}^Z)$ with respect to Lebesgue measure. In this case, the Fourier transform of these functions exists, and we will write

$$\widehat{\partial_z V}(r; Z) = \frac{1}{(2\pi)^{|Z|}} \int_{\mathbb{R}^Z} e^{-ir \cdot q} \, \partial_z V(q; Z) \, dq, \tag{4.6}$$

for any $r \in \mathbb{R}^Z$.
(vi) For each $Z \subset \Lambda_L$, we assume that the Fourier inversion formula holds for all first order partial derivatives of V_Z. Thus, for any $q \in \mathbb{R}^Z$,

$$\partial_z V(q; Z) = \int_{\mathbb{R}^Z} e^{ir \cdot q} \, \widehat{\partial_z V}(r; Z) \, dr, \tag{4.7}$$

and therefore, we will write

$$\left[\partial_z V_Z \right] (\mathrm{x}) = \int_{\mathbb{R}^Z} \left[W(r \cdot \delta_Z) \right] (\mathrm{x}) \, \widehat{\partial_z V}(r; Z) \, dr, \tag{4.8}$$

where the function $r \cdot \delta_Z : \Lambda_L \to \mathbb{R}$ is given by

$$[r \cdot \delta_Z](x) = \sum_{z \in Z} r_z \, \delta_z(x) \quad \text{hence } \left[W(r \cdot \delta_Z) \right] (\mathrm{x}) = \exp \left[i \sum_{z \in Z} r_z q_z \right] , \tag{4.9}$$

as required.

(vii) There exist numbers $C_3 \geq 0$ and $\mu_3 \geq 0$ such that given any points $x, y \in \Lambda_L$, the bound

$$\sum_{\substack{Z \subset \Lambda_L \\ x,y \in Z}} \int_{\mathbb{R}^Z} |r| \cdot \left| \widehat{\nabla V}(r; Z) \right| dr \leq C_3 \, F_{\mu_3}(d(x, y)) . \tag{4.10}$$

Here the vector-valued function $\widehat{\nabla V}(\cdot; Z) : \mathbb{R}^Z \to \mathbb{C}^Z$ has components $\widehat{\partial_z V}(\cdot; Z)$ for each $z \in Z$. The number $|r|$, corresponding to some $r \in \mathbb{R}^Z$, is taken as $|r| = \sum_{z \in Z} |r_z|$, but, as is seen in the proof below, any norm on $\mathbb{R}^Z$ satisfying $|r_z| \leq \|r\|$ will suffice. As will become apparent below, we interpret the function F_{μ_3} in assumption (vii) as our crucial estimate on the range of the interactions.

We now state our most general result.

Theorem 8 *Let X and Y be finite subsets of $\mathbb{Z}^\nu$ and take $L_0 \geq 1$ to be the smallest integer such that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$ and $t \in \mathbb{R}$, let α_t^L denote the dynamics corresponding to the anharmonic Hamiltonian H^{Λ_L} in* (4.3), *and suppose that the perturbation satisfies assumptions* (i)–(vii) *listed above. Then, for each $\epsilon > 0$ and any functions $f, g \colon \Lambda_{L_0} \to \mathbb{C}$ with the support of f in X and the support of g in Y,*

$$\left\| \left\{ \alpha_t^L(W(f)), W(g) \right\} \right\|_\infty \leq C \, \|f\|_\infty \|g\|_\infty \, e^{\delta |t|} \sum_{x \in X, y \in Y} F_{\mu_3}(d(x, y)) \tag{4.11}$$

holds for any $t \in \mathbb{R}$. Here

$$C = (1 + c_{\omega,\lambda} e^{\frac{(\mu_3+\epsilon)}{2}} + c_{\omega,\lambda}^{-1}) \sup_{s \geq 0} \left[(1+s)^{\nu+1} e^{-\epsilon s} \right] \tag{4.12}$$

and

$$\delta = \delta(\epsilon) = (\mu_3 + \epsilon) \, v_h(\mu_3 + \epsilon) + C \, C_3 \, C_\nu^2, \tag{4.13}$$

where v_h is as in (2.30), *C_ν is in* (2.47), *and C_3 is in* (4.10).

One important difference between the bound we prove in Theorem 8 above, in contrast to the one proven in Theorem 7, is that the spatial decay rate in (4.11) can be no greater than the rate μ_3 appearing in (4.10). If $\mu_3 > 0$, then there is a corresponding velocity for this anharmonic system

$$v_{ah}(\epsilon) = \left(1 + \frac{\epsilon}{\mu_3} \right) v_h(\mu_3 + \epsilon) + \frac{C \, C_3 \, C_\nu^2}{\mu_3}. \tag{4.14}$$

Since the case of $\mu_3 = 0$ represents only polynomial decay in the interaction range, as measured by (4.10), the bound in (4.11) at most decays polynomially in distance between the supports of f and g as well.

Example 1 To clarify the general assumptions on the perturbation introduced above, we will consider a simple model with pair interactions generated by a single function. One can compare this example with the on-site, anharmonic Hamiltonian analyzed in Sect. 3. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be given and fix some number $\mu \geq 0$. For each $L \geq 1$ and any $Z \subset \Lambda_L$, define

$$V(\cdot;Z)=\begin{cases}F_\mu(d(z_1,z_2))V(\cdot) & \text{if } Z=\{z_1,z_2\},\\ 0 & \text{otherwise,}\end{cases} \tag{4.15}$$

and thereby, the anharmonic Hamiltonian

$$H^{\Lambda_L}=H_h^{\Lambda_L}+\sum_{z_1,z_2\in\Lambda_L}V_{\{z_1,z_2\}}, \tag{4.16}$$

with $V_{\{z_1,z_2\}}(\mathrm{x})=F_\mu\left(d(z_1,z_2)\right)\cdot V(q_{z_1},q_{z_2})$. As one can easily check, the basic assumptions (i)–(iv) follow if V has well-defined, second order partial derivatives and there exist numbers C_1, $\tilde{C}_1$, and C_2 such that

$$\max_{i=1,2}|\partial_i V(x,y)|\leq C_1\left(|x|+|y|+\tilde{C}_1\right) \tag{4.17}$$

and

$$\max_{i,j\in\{1,2\}}\left|\partial_i\partial_j V(x,y)\right|\leq C_2. \tag{4.18}$$

If both first order partial derivatives of V are integrable and satisfy the Fourier inversion formula, then the condition (vii) is satisfied when

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(|x|+|y|)\left(\left|\widehat{\partial_1 V}(x,y)\right|+\left|\widehat{\partial_2 V}(x,y)\right|\right)dx\,dy<\infty. \tag{4.19}$$

Thus, under the above conditions, the model described by (4.16) satisfies the assumptions of Theorem 8, and hence the corresponding locality result (4.11) is valid.

Proof of Theorem 8 Much of the argument in the proof of Theorem 7 also applies here. Again, we fix L, regard both Hamiltonians on the same volume, drop the dependence of each of the dynamics on L, and interpolate. Let $t>0$ and set

$$\Phi_t(s)=\{\alpha_s(\alpha_{t-s}^h(W(f))),W(g)\} \tag{4.20}$$

for $0\leq s\leq t$. The calculation

$$\begin{aligned}\frac{d}{ds}\alpha_s(\alpha_{t-s}^h(W(f)))&=\sum_{Z\subset\Lambda_L}\alpha_s\left(\left\{\alpha_{t-s}^h(W(f)),V_Z\right\}\right)\\&=-i\sum_{Z\subset\Lambda_L}\sum_{z\in Z}\mathrm{Im}\left[f_{t-s}(z)\right]\alpha_s\left(W(f_{t-s})\right)\cdot\alpha_s\left(\partial_z V_Z\right),\end{aligned} \tag{4.21}$$

follows readily, and therefore, we derive a differential equation analogous to (3.12); namely

$$\frac{d}{ds}\Phi_t(s)=i\tilde{\mathcal{L}}_t(s)\Phi_t(s)+\tilde{\mathcal{Q}}_t(s), \tag{4.22}$$

where

$$\begin{aligned}\tilde{\mathcal{L}}_t(s)&=-\sum_{Z\subset\Lambda_L}\sum_{z\in Z}\mathrm{Im}\left[f_{t-s}(z)\right]\alpha_s(\partial_z V_Z),\\ \tilde{\mathcal{Q}}_t(s)&=-i\sum_{Z\subset\Lambda_L}\sum_{z\in Z}\mathrm{Im}\left[f_{t-s}(z)\right]\alpha_s\left(\alpha_{t-s}^h(W(f))\right)\{\alpha_s(\partial_z V_Z),W(g)\}.\end{aligned} \tag{4.23}$$

Arguing as before, we arrive at the bound

$$\|\{\alpha_t(W(f)), W(g)\}\|_\infty \leq \left\| \left\{ \alpha_t^h(W(f)), W(g) \right\} \right\|_\infty + \sum_{Z \subset \Lambda_L} \sum_{z \in Z} \int_0^t \left| \mathrm{Im}\left[f_{t-s}(z) \right] \right| \|\{\alpha_s(\partial_z V_Z), W(g)\}\|_\infty \, ds. \tag{4.24}$$

Inserting (4.8) into (4.24) and using the harmonic bounds from Corollary 1 with $\mu = \mu_3$, we find that

$$\begin{aligned} &\|\{\alpha_t(W(f)), W(g)\}\|_\infty \\ &\leq C \|f\|_\infty \|g\|_\infty e^{\hat{v}t} \sum_{x \in X, y \in Y} F_{\mu_3}(d(x, y)) \\ &\quad + C \|f\|_\infty \sum_{Z \subset \Lambda_L} \sum_{z \in Z} \sum_{x \in X} F_{\mu_3}(d(x, z)) \\ &\quad \times \int_0^t e^{\hat{v}(t-s)} \int_{\mathbb{R}^Z} |\widehat{\partial_z V}(r; Z)| \left\| \{\alpha_s \left(W(r \cdot \delta_Z) \right), W(g)\} \right\|_\infty dr \, ds, \end{aligned} \tag{4.25}$$

with C as in (4.12), and we have set $\hat{v} = (\mu_3 + \epsilon) v_h(\mu_3 + \epsilon)$ for notational convenience.

After iterating (4.25) $m \geq 1$ times, we find that

$$\|\{\alpha_t(W(f)), W(g)\}\|_\infty \leq C \|f\|_\infty \|g\|_\infty e^{\hat{v}t} \sum_{x \in X, y \in Y} \sum_{n=0}^{m} \tilde{a}_n(x, y; t) + \tilde{R}_{m+1}(t), \tag{4.26}$$

where

$$\tilde{a}_0(x, y; t) = F_{\mu_3}(d(x, y)), \tag{4.27}$$

$$\begin{aligned} \tilde{a}_1(x, y; t) &= Ct \sum_{Z \subset \Lambda_L} \sum_{z_1, z_2 \in Z} \int_{\mathbb{R}^Z} \|r \cdot \delta_Z\|_\infty \cdot |\widehat{\partial_{z_1} V}(r; Z)| \, dr \, F_{\mu_3}\left(d(x, z_1)\right) F_{\mu_3}\left(d(z_2, y)\right) \\ &\leq Ct \sum_{z_1, z_2 \in \Lambda_L} F_{\mu_3}\left(d(x, z_1)\right) F_{\mu_3}\left(d(z_2, y)\right) \sum_{\substack{Z \subset \Lambda_L \\ z_1, z_2 \in Z}} \int_{\mathbb{R}^Z} |r| \cdot |\widehat{\nabla V}(r; Z)| \, dr \\ &\leq C C_3 t \sum_{z_1, z_2 \in \Lambda_L} F_{\mu_3}\left(d(x, z_1)\right) F_{\mu_3}\left(d(z_1, z_2)\right) F_{\mu_3}\left(d(z_2, y)\right) \\ &\leq C C_3 C_\nu^2 t F_{\mu_3}\left(d(x, y)\right), \end{aligned} \tag{4.28}$$

and in general,

$$\begin{aligned} &\tilde{a}_n(x, y; t) \\ &= \frac{(Ct)^n}{n!} \sum_{Z_1, Z_2, \ldots, Z_n \subset \Lambda_L} \sum_{z_{1,1}, z_{1,2} \in Z_1} \sum_{z_{2,1}, z_{2,2} \in Z_2} \cdots \\ &\quad \times \sum_{z_{n,1}, z_{n,2} \in Z_n} \left(\prod_{j=1}^{n} \int_{\mathbb{R}^{Z_j}} \|r_j \cdot \delta_{Z_j}\|_\infty \left| \widehat{\partial_{z_{j,1}} V}(r_j; Z_j) \right| dr_j \right) \end{aligned}$$

$$
\begin{aligned}
&\times F_{\mu_3}\left(d(x, z_{1,1})\right) \cdot F_{\mu_3}\left(d(z_{1,2}, z_{2,1})\right) \cdots F_{\mu_3}\left(d(z_{n,2}, y)\right) \\
&\leq \frac{(Ct)^n}{n!} \sum_{z_{1,1}, z_{1,2} \in \Lambda_L} \sum_{z_{2,1}, z_{2,2} \in \Lambda_L} \cdots \sum_{z_{n,1}, z_{n,2} \in \Lambda_L} F_{\mu_3}\left(d(x, z_{1,1})\right) \\
&\times F_{\mu_3}\left(d(z_{1,2}, z_{2,1})\right) \cdots F_{\mu_3}\left(d(z_{n,2}, y)\right) \\
&\times \prod_{j=1}^{n} \sum_{\substack{Z_j \subset \Lambda_L \\ z_{j,1}, z_{j,2} \in Z_j}} \int_{\mathbb{R}^{Z_j}} |r_j| \cdot |\widehat{\nabla V}(r_j; Z_j)| \, dr_j \\
&\leq \frac{(C\, C_3\, t)^n}{n!} \sum_{z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2}, \ldots, z_{n,1}, z_{n,2} \in \Lambda_L} F_{\mu_3}\left(d(x, z_{1,1})\right) \cdot F_{\mu_3}\left(d(z_{1,1}, z_{1,2})\right) \\
&\times F_{\mu_3}\left(d(z_{1,2}, z_{2,1})\right) \cdots F_{\mu_3}\left(d(z_{n,2}, y)\right) \\
&\leq \frac{(C\, C_3\, C_\nu^2\, t)^n}{n!} F_{\mu_3}\left(d(x, y)\right),
\end{aligned} \tag{4.29}
$$

for any $1 \leq n \leq m$. As before, with $t > 0$ fixed, the remainder term $\tilde{R}_{m+1}(t)$ converges to zero as $m \to \infty$. Thus we have proven (4.11) as claimed. □

5 A Priori Solution Estimates

In this section, we will prove a variety of a priori estimates which will be useful in our proofs of the main results. The underlying argument which facilitates most of the lemmas below is the well-known Gronwall inequality. We state and prove a version of this estimate which is tailored to the present work. A more general bound of this type appears, e.g. in [1].

Lemma 3 (Gronwall Inequality) *Let $u : \mathbb{R} \to \mathbb{C}$ satisfy*

$$
|u(t)| \leq \alpha(t) + \int_a^t f(t, s)\, |u(s)| \, ds \tag{5.1}
$$

for all t in $[a, b]$. If α is non-negative and non-decreasing and f is non-negative and continuous with $f(\cdot, s)$ nondecreasing for each fixed $s \in [a, b]$, then

$$
|u(t)| \leq \alpha(t) \exp\left(\int_a^t f(t, s)\, ds\right) \tag{5.2}
$$

for all t in $[a, b]$.

Proof We prove (5.2) pointwise. Let $t_0 \in [a, b]$ and observe that

$$
|u(t)| \leq \alpha(t_0) + \int_a^t f(t_0, s)\, |u(s)| \, ds, \tag{5.3}
$$

holds for all $t \in [a, t_0]$. Define

$$
m(t) = \alpha(t_0) + \int_a^t f(t_0, s) |u(s)| \, ds. \tag{5.4}
$$

Clearly, $|u(t)| \leq m(t)$ and the bound

$$m'(t) = f(t_0, t)\,|u(t)| \leq f(t_0, t)m(t), \tag{5.5}$$

readily implies

$$|u(t)| \leq m(t) \leq m(0) \exp\left(\int_a^t f(t_0, s)\, ds \right), \tag{5.6}$$

for any $t \in [a, t_0]$. Taking $t = t_0$, we have proven (5.2). $\square$

The applications we have in mind concern bounding the solutions of our Hamiltonian flows. Recall that our general, finite volume, multi-site Hamiltonian, $H^{\Lambda_L} : \mathcal{X}_{\Lambda_L} \to \mathbb{R}$, is of the form

$$H^{\Lambda_L} = H_h^{\Lambda_L} + \sum_{Z \subset \Lambda_L} V_Z, \tag{5.7}$$

and we need a variety of assumptions on the perturbations V_Z to prove our estimates.

We begin with a basic proof of boundedness for the flow $\Phi_t : \mathcal{X}_{\Lambda_L} \to \mathcal{X}_{\Lambda_L}$ corresponding to (5.7). As is demonstrated in [9], boundedness follows if the perturbation is dominated by the harmonic part. For the sake of completeness, we include this argument here.

We assume the perturbation in (5.7) above satisfies: There exist numbers $C_1 \geq 0$, $\tilde{C}_1 \geq 0$, and $\mu_1 \geq 0$ such that

$$\left(\sum_{Z \subset \Lambda_L} |\partial_x V_Z(\mathrm{x})| \right)^2 \leq C_1 \sum_{y \in \Lambda_L} (q_y^2 + \tilde{C}_1) F_{\mu_1}(d(x, y)), \tag{5.8}$$

for each $x \in \Lambda_L$ and any $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$.

Lemma 4 *Fix $L \geq 1$ and let Φ_t denote the flow corresponding to the Hamiltonian H^{Λ_L} defined in (5.7) above. If the perturbation satisfies (5.8) described above, then for any $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$ the components of the flow $\Phi_t(\mathrm{x}) = \{(q_x(t), p_x(t))\}_{x \in \Lambda_L}$ satisfy*

$$\sup_{x \in \Lambda_L} \max\left(|q_x(t)|, |p_x(t)|\right) \leq K_1 \exp(K_2 t), \tag{5.9}$$

where

$$K_1 = K_1(\mathrm{x}) = \sqrt{\sup_{x \in \Lambda_L} \left(p_x^2(0) + q_x^2(0) + \tilde{C}_1 \right)}, \tag{5.10}$$

and

$$K_2 = \left| \omega^2 + 2\sum_{j=1}^{\nu} \lambda_j - 1 \right| + 4\sum_{j=1}^{\nu} \lambda_j + \frac{1}{2} + \frac{C_1}{2} \sum_{x \in \Lambda_L} F_{\mu_1}(d(0, x)). \tag{5.11}$$

Proof Fix $L \geq 1$, take $x \in \Lambda_L$, and choose $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$. Consider the function defined by setting

$$E_x(t) = p_x^2(t) + q_x^2(t) + \tilde{C}_1, \tag{5.12}$$

where $\tilde{C}_1 > 0$ is the number appearing in (5.8). From Hamilton's equations, we have that

$$\begin{aligned}
\dot{E}_x(t) &= 2p_x(t)\dot{p}_x(t) + 2q_x(t)\dot{q}_x(t) \\
&= -4\left(\omega^2 + 2\sum_{j=1}^{\nu}\lambda_j - 1\right) p_x(t)\, q_x(t) + 4p_x(t)\sum_{j=1}^{\nu}\lambda_j\left(q_{x+e_j}(t) + q_{x-e_j}(t)\right) \\
&\quad - 2p_x(t)\sum_{Z\subset\Lambda_L}[\partial_x V_Z](\Phi_t(\mathrm{x})), \qquad (5.13)
\end{aligned}$$

and therefore,

$$\left|\dot{E}_x(t)\right| \leq \sum_{y\in\Lambda_L} A_{x,y}E_y(t) = (AE(t))_x \qquad (5.14)$$

where the $|\Lambda_L| \times |\Lambda_L|$ matrix $A = (A_{x,y})$ is given by

$$A_{x,y} = \begin{cases} 2|\omega^2 + 2\sum_{j=1}^{\nu}\lambda_j - 1| + 4\sum_{j=1}^{\nu}\lambda_j + 1 + C_1\, F_{\mu_1}(0) & \text{if } y = x, \\ 2\lambda_j + C_1\, F_{\mu_1}(1) & \text{if } y = x \pm e_j, \\ C_1\, F_{\mu_1}(d(x,y)) & \text{otherwise,} \end{cases} \qquad (5.15)$$

and $(AE(t))_x$ is the x-th component of this vector. Denote by E the vector-valued function whose components are E_x and equip $\mathbb{R}^{|\Lambda_L|}$ with the sup-norm $\|\cdot\|_\infty$. With (5.14) it is clear that

$$\left\|\dot{E}(t)\right\|_\infty \leq \|AE(t)\|_\infty, \qquad (5.16)$$

and therefore,

$$\|E(t)\|_\infty \leq \|E(0)\|_\infty + \int_0^t \left\|\dot{E}(s)\right\|_\infty ds \leq \|E(0)\|_\infty + \int_0^t \|AE(s)\|_\infty\, ds. \qquad (5.17)$$

Letting $u(t) = \|E(t)\|_\infty$, Lemma 3 implies

$$\max\left(|q_x(t)|^2, |p_x(t)|^2\right) \leq \|E(t)\|_\infty \leq \|E(0)\|_\infty \exp\left(\|A\|_\infty t\right), \qquad (5.18)$$

from which (5.9) is clear. □

As will become clear in the proof of Lemma 6 below, the main quantities of interest for us are the derivatives of the components of the flow with respect to the initial conditions. The next lemma provides explicit estimates on these functions. To prove it we need the following additional assumption on our perturbation. Assume there exists constants $C_2 \geq 0$ and $\mu_2 \geq 0$ for which: given any $L \geq 1$ and any pair $x, y \in \Lambda_L$, the bound

$$\sum_{Z\subset\Lambda_L}\left|\left[\partial_x\partial_y V_Z\right](\mathrm{x})\right| \leq C_2 F_{\mu_2}(d(x,y)), \qquad (5.19)$$

holds for all points $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$.

Lemma 5 *Fix $L \geq 1$ and let Φ_t denote the flow corresponding to the Hamiltonian H^{Λ_L} defined in* (5.7) *above. If the perturbation satisfies* (5.8) *and* (5.19), *then for any* $\mathrm{x} \in \mathcal{X}_{\Lambda_L}$ *the* *components of the flow* $\Phi_t(\mathrm{x}) = \{(q_x(t), p_x(t))\}_{x\in\Lambda_L}$ *satisfy*

$$\sup_{x,y\in\Lambda_L} \max\left(\left|\frac{\partial q_x(t)}{\partial q_y(0)}\right|, \left|\frac{\partial q_x(t)}{\partial p_y(0)}\right|\right) \leq \max(1, 2t)\exp(K\,t^2), \tag{5.20}$$

and

$$\sup_{x,y\in\Lambda_L} \max\left(\left|\frac{\partial p_x(t)}{\partial q_y(0)}\right|, \left|\frac{\partial p_x(t)}{\partial p_y(0)}\right|\right) \leq 1 + t\left(K + 2\sum_{j=1}^{\nu}\lambda_j\right)\max(1, 2t)\exp(K\,t^2), \tag{5.21}$$

where

$$K = 2\omega^2 + 8\sum_{j=1}^{\nu}\lambda_j + C_2\sum_{x\in\Lambda_L} F_{\mu_2}(d(0,x)). \tag{5.22}$$

Proof We begin with a proof of (5.20). Using Hamilton's equations, we find that

$$\begin{aligned} q_x(t) - q_x(0) - 2t\,p_x(0) &= 2\int_0^t\int_0^s \dot{p}_x(r)\,dr\,ds \\ &= -4\left(\omega^2 + 2\sum_{j=1}^{\nu}\lambda_j\right)\int_0^t (t-s)\,q_x(s)\,ds \\ &\quad + 4\sum_{j=1}^{\nu}\lambda_j\int_0^t (t-s)\left(q_{x+e_j}(s) + q_{x-e_j}(s)\right)ds \\ &\quad - 2\sum_{Z\subset\Lambda_L}\int_0^t (t-s)\,[\partial_x V_Z](\Phi_s(\mathrm{x}))\,ds \end{aligned} \tag{5.23}$$

and therefore

$$\begin{aligned} \frac{\partial q_x(t)}{\partial q_y(0)} = \delta_x(y) &- 4\left(\omega^2 + 2\sum_{j=1}^{\nu}\lambda_j\right)\int_0^t (t-s)\,\frac{\partial q_x(s)}{\partial q_y(0)}\,ds \\ &+ 4\sum_{j=1}^{\nu}\lambda_j\int_0^t (t-s)\left(\frac{\partial q_{x+e_j}(s)}{\partial q_y(0)} + \frac{\partial q_{x-e_j}(s)}{\partial q_y(0)}\right)ds \\ &- 2\sum_{z\in\Lambda_L}\sum_{Z\subset\Lambda_L}\int_0^t (t-s)\left[\partial_z\partial_x V_Z\right](\Phi_s(\mathrm{x}))\cdot\frac{\partial q_z(s)}{\partial q_y(0)}\,ds. \end{aligned} \tag{5.24}$$

More succinctly, we have found that

$$\left|\frac{\partial q_x(t)}{\partial q_y(0)}\right| \leq \delta_x(y) + \sum_{z\in\Lambda_L}\int_0^t (t-s)A_{x,z}\left|\frac{\partial q_z(s)}{\partial q_y(0)}\right| ds \tag{5.25}$$

where the $|\Lambda_L| \times |\Lambda_L|$ matrix $A = (A_{x,z})$ has entries

$$A_{x,z} = \begin{cases} 4(\omega^2 + 2\sum_{j=1}^{\nu} \lambda_j) + 2C_2 F_{\mu_2}(0) & \text{if } z = x, \\ 4\lambda_j + 2C_2 F_{\mu_2}(1) & \text{if } z = x \pm e_j, \\ 2C_2 F_{\mu_2}(d(x,z)) & \text{otherwise.} \end{cases} \tag{5.26}$$

If, for each fixed $y \in \Lambda_L$, we denote by $\partial_y q : \mathbb{R} \to \mathbb{R}^{|\Lambda_L|}$ the vector-valued function whose components are given by $|\frac{\partial q_x(t)}{\partial q_y(0)}|$, then (5.25) implies

$$\left\| \partial_y q(t) \right\|_\infty \leq 1 + \int_0^t (t-s) \left\| A \partial_y q(s) \right\|_\infty ds. \tag{5.27}$$

Again letting $u(t) = \|\partial_y q(t)\|_\infty$, Lemma 3 yields the estimate

$$\left\| \partial_y q(t) \right\|_\infty \leq \exp\left(\frac{\|A\|_\infty t^2}{2} \right). \tag{5.28}$$

Quite similarly, the bound

$$\left| \frac{\partial q_x(t)}{\partial p_y(0)} \right| \leq 2t\, \delta_x(y) + \sum_{z \in \Lambda_L} \int_0^t (t-s) A_{x,z} \left| \frac{\partial q_z(s)}{\partial q_y(0)} \right| ds, \tag{5.29}$$

follows from (5.23) with the same matrix A. This proves (5.20).

The bound for $p_x(t)$ follows from (5.20). In fact, it is easy to see that

$$p_x(t) = p_x(0) - 2\left(\omega^2 + 2\sum_{j=1}^{\nu} \lambda_j \right) \int_0^t q_x(s)\, ds$$
$$+ 2\sum_{j=1}^{\nu} \lambda_j \int_0^t \left(q_{x+e_j}(s) + q_{x-e_j}(s) \right) ds \; - \sum_{Z \subset \Lambda_L} \int_0^t [\partial_x V_Z](\Phi_s(\mathrm{x}))\, ds. \tag{5.30}$$

Using (5.20), (5.21) readily follows. □

Lemma 6 *Let $X, Y \subset \mathbb{Z}^\nu$ be finite sets and take $L_0 \geq 1$ large enough so that $X, Y \subset \Lambda_{L_0}$. For any $L \geq L_0$ and $t \in \mathbb{R}$, denote by α_t^L the dynamics corresponding to (5.7). If the perturbation satisfies (5.8) and (5.19), then there exist positive numbers K_1 and K_2, both independent of L, for which: given any functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$, the bound*

$$\left\| \left\{ \alpha_t^L(W(f)), W(g) \right\} \right\|_\infty \leq K_1 |X| |Y| \|f\|_\infty \|g\|_\infty \exp(K_2 t^2), \tag{5.31}$$

holds for all $t \in \mathbb{R}$.

Proof We first fix $L \geq L_0$ as in the statement of the lemma and prove the estimate on Λ_L. In this case, we suppress the dependence of most quantities on L to ease notation. Now, recall that for any fixed point x,

$$\left[\{\alpha_t(W(f)), W(g)\}\right](\mathrm{x}) = \sum_{y\in\Lambda_L} \frac{\partial}{\partial q_y}\left[\alpha_t(W(f))\right](\mathrm{x}) \cdot \frac{\partial}{\partial p_y}\left[W(g)\right](\mathrm{x}) - \sum_{y\in\Lambda_L} \frac{\partial}{\partial p_y}\left[\alpha_t(W(f))\right](\mathrm{x}) \cdot \frac{\partial}{\partial q_y}\left[W(g)\right](\mathrm{x}). \quad (5.32)$$

Since

$$\begin{aligned} \frac{\partial}{\partial p_y}\left[W(g)\right](\mathrm{x}) &= i\,\mathrm{Im}\left[g(y)\right]\left[W(g)\right](\mathrm{x}) \quad \text{and} \\ \frac{\partial}{\partial q_y}\left[W(g)\right](\mathrm{x}) &= i\,\mathrm{Re}\left[g(y)\right]\left[W(g)\right](\mathrm{x}), \end{aligned} \quad (5.33)$$

the sums in (5.32) above are only over those y in the support of g. The derivative of the time-evolved quantities may also be calculated, e.g.,

$$\frac{\partial}{\partial q_y}\left[\alpha_t(W(f))\right](\mathrm{x}) = i\left(\sum_{x\in\Lambda_L} \mathrm{Re}\left[f(x)\right]\cdot\frac{\partial q_x(t)}{\partial q_y} + \mathrm{Im}\left[f(x)\right]\cdot\frac{\partial p_x(t)}{\partial q_y}\right)\cdot\left[\alpha_t(W(f))\right](\mathrm{x}). \quad (5.34)$$

Clearly, the sum in (5.34) is only over those x in the support of f, and a similar formula holds for the derivative with respect to p_y. Thus,

$$\begin{aligned} &\left|\left[\{\alpha_t(W(f)), W(g)\}\right](\mathrm{x})\right| \\ &\quad \le 4\|f\|_\infty\|g\|_\infty \sum_{x\in X, y\in Y} \max\left(\left|\frac{\partial q_x(t)}{\partial q_y}\right|, \left|\frac{\partial p_x(t)}{\partial q_y}\right|, \left|\frac{\partial q_x(t)}{\partial p_y}\right|, \left|\frac{\partial p_x(t)}{\partial p_y}\right|\right). \end{aligned}$$

Using Lemma 5, (5.31) immediately follows. □

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